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# Boundedness of $\gamma$ -Cesàro means ( $\gamma > 0$ ) of operators

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In this talk I would like to report some recent results on the boundedness properties of  $\gamma$ -Cesàro means of operators, where  $\gamma > 0$ . The results are taken from joint works with Jeng-Chung Chen, Yuan-Chuan Li, and Sen-Yen Shaw (cf. [1], [3]).

**1. The discrete case.** Let  $T : X \rightarrow X$  be a bounded linear operator on a Banach space  $X$ . The Cesàro means of order  $\gamma$  (or  $\gamma$ -Cesàro means) of  $T$ , where  $\gamma \geq 0$ , are defined by

$$C_n^\gamma = C_n^\gamma(T) := \frac{1}{\sigma_n^\gamma} \sum_{k=0}^n \sigma_{n-k}^{\gamma-1} T^k, \quad n = 0, 1, 2, \dots,$$

where  $\sigma_n^\beta = \binom{\beta+n}{n}$  for  $n \geq 1$ , and  $\sigma_0^\beta = 1$  (see Zygmund [5, Chapter 3]). Among them are the following two particular means:  $C_n^0 = C_n^0(T) = T^n$ , and  $C_n^1 = C_n^1(T) = (n+1)^{-1} \sum_{k=0}^n T^k$ .

The Abel means of  $T$  are the operators  $A_r = A_r(T) := (1-r) \sum_{k=0}^\infty r^k T^k$ , defined for  $0 < r < 1/r(T)$ , where  $r(T) = \lim_{n \rightarrow \infty} \|T^n\|^{1/n}$  denotes the spectral radius of  $T$ . Clearly,  $r(T) \leq 1$  if and only if  $A_r$  exists for all  $0 < r < 1$ . (Moreover, in this case, we have  $A_r = (1-r)(I - rT)^{-1}$  for each  $0 < r < 1$ .) The following is well-known (cf. [5]):

If  $0 < \gamma < \beta < \infty$ , then

$$(1) \quad \sup_{0 < r < 1} \|A_r\| \leq \sup_{n \geq 0} \|C_n^\beta\| \leq \sup_{n \geq 0} \|C_n^\gamma\| \leq \sup_{n \geq 0} \|C_n^0\| (= \sup_{n \geq 0} \|T^n\|);$$

in particular, if  $T$  is a positive linear operator on a Banach lattice  $X$ , then

$$(2) \quad \sup_{0 < r < 1} \|A_r\| < \infty \iff \sup_{n \geq 0} \|C_n^1\| < \infty \quad (\text{cf. Emilion [2]}).$$

In connection with these relations, two questions come up naturally:

(A) If  $T$  is positive, then does the implication " $\sup_{0 < r < 1} \|A_r\| < \infty \Rightarrow \sup_{n \geq 1} \|C_n^\gamma\| < \infty$ " hold for a certain constant  $\gamma$ , with  $0 < \gamma < 1$ ?

(B) If  $T$  is not assumed to be positive, then does the implication " $\sup_{0 < r < 1} \|A_r\| < \infty \Rightarrow \sup_{n \geq 1} \|C_n^\gamma\| < \infty$ " hold for a certain constant  $\gamma$ , with  $\gamma \geq 1$ ?

Our answers are as follows.

**Theorem 1.** For any  $\gamma$ , with  $0 < \gamma < 1$ , there exists a positive linear operator  $T$  on an  $L_1$ -space such that  $\sup_{n \geq 1} \|C_n^\beta\| < \infty$  for all  $\beta > \gamma$ , but  $\sup_{n \geq 1} \|C_n^\gamma\| = \infty$ .

**Theorem 2.** There exists a positive linear operator  $T$  on an  $L_1$ -space such that  $\sup_{n \geq 1} \|C_n^\beta\| < \infty$  for all  $\beta > 0$ , but  $\sup_{n \geq 1} \|T^n\| = \infty$ .

**Theorem 3.** Let  $\dim X = \infty$ . Then the following hold:

(i) For any integer  $k \geq 1$ , there exists a bounded linear operator  $T$  on  $X$  such that  $\sup_{n \geq 1} \|C_n^k\| < \infty$ , but  $\sup_{n \geq 1} \|C_n^\beta\| = \infty$  for all  $\beta$ , with  $0 \leq \beta < k$ .

(ii) There exists a bounded linear operator  $T$  on  $X$ , with  $r(T) = 1$ , such that  $\sup_{0 < r < 1} \|A_r\| < \infty$ , but  $\sup_{n \geq 1} \|C_n^\beta\| = \infty$  for all  $\beta \geq 0$ .

**2. The continuous case.** Let  $T(\cdot)$  be a  $C_0$ -semigroup of bounded linear operators on a Banach space  $X$ . The  $\gamma$ -th Cesàro means of  $T(\cdot)$ , where  $\gamma \geq 0$ , are defined as  $C_0^\gamma = C_0^\gamma(T(\cdot)) := T(0)$  and, for  $t > 0$ ,

$$C_t^\gamma = C_t^\gamma(T(\cdot)) := \begin{cases} T(t) & \text{if } \gamma = 0, \\ \gamma t^{-\gamma} \int_0^t (t-u)^{\gamma-1} T(u) du & \text{if } \gamma > 0. \end{cases}$$

The Abel means of  $T(\cdot)$  are the operators

$$A_\lambda = A_\lambda(T(\cdot)) := \lambda \int_0^\infty e^{-\lambda u} T(u) du = \lim_{t \rightarrow \infty} \lambda \int_0^t e^{-\lambda u} T(u) du,$$

defined for  $\lambda > 0$  if the limit exists. As in the discrete case, we have (cf. [4]):

If  $0 < \gamma < \beta < \infty$ , then

$$(3) \quad \sup_{0 < \lambda < \infty} \|A_\lambda\| \leq \sup_{t > 0} \|C_t^\beta\| \leq \sup_{t > 0} \|C_t^\gamma\| \leq \sup_{t > 0} \|C_t^0\| (= \sup_{t > 0} \|T(t)\|);$$

inparticular, if  $T(\cdot)$  is a positive  $C_0$ -semigroup on a Banach lattice  $X$ , then

$$(4) \quad \sup_{0 < \lambda < \infty} \|A_\lambda\| < \infty \iff \sup_{t > 0} \|C_t^1\| < \infty \quad (\text{cf. [2]}).$$

The following are the continuous case results:

**Theorem 1'.** *For any  $\gamma$ , with  $0 < \gamma < 1$ , there exists a positive  $C_0$ -semigroup  $T(\cdot)$  on an  $L_1$ -space such that  $\sup_{t > 0} \|C_t^\beta\| < \infty$  for all  $\beta > \gamma$ , but  $\sup_{t > 0} \|C_t^\gamma\| = \infty$ .*

**Theorem 2'.** *There exists a positive  $C_0$ -semigroup  $T(\cdot)$  on an  $L_1$ -space such that  $\sup_{t > 0} \|C_t^\beta\| < \infty$  for all  $\beta > 0$ , but  $\sup_{t > 0} \|T(t)\| = \infty$ .*

**Theorem 3'.** *Let  $\dim X = \infty$ . Then the following hold:*

(i) *For any integer  $k > 1$ , there exists a  $C_0$ -semigroup  $T(\cdot)$  on  $X$  such that  $\sup_{t > 0} \|C_t^k\| < \infty$ , but  $\sup_{t > 0} \|C_t^\beta\| = \infty$  for all  $\beta$ , with  $0 \leq \beta < k$ .*

(ii) *There exists a  $C_0$ -semigroup  $T(\cdot)$  on  $X$  such that  $\sup_{0 < \lambda < \infty} \|A_\lambda\| < \infty$ , but  $\sup_{t > 0} \|C_t^\beta\| = \infty$  for all  $\beta \geq 0$ .*

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